

Analysis of a 3D chaotic system

Gheorghe Tigan ^{*}, Dumitru Opris [†]

Abstract

A 3D nonlinear chaotic system, called the T system, is analyzed in this paper. Horseshoe chaos is investigated via the heteroclinic Shilnikov method constructing a heteroclinic connections between the saddle equilibrium points of the system. Partially numerical computations are carried out to support the analytical results.

1 Introduction

One of the first examples of continuous dynamical system of dimension three, whose numerical simulations display the property of sensitivity to initial conditions, is the Lorenz system [5]:

$$\dot{x} = a(y - x), \quad \dot{y} = cx - y - xz, \quad \dot{z} = -bz + xy, \quad (1)$$

where a, b, c are positive real parameters. Physical realization of the Lorenz system is the Rayleigh-Benard experiment. The system was derived from the hydrodynamical Navier-Stokes equations to create a dynamical model for meteorology.

Trying to transform the Lorenz system from a stable state to a chaotic one (concept known as anticontrol of chaos or chaotification, [4]), Chen in [3] introduces a 3D polynomial system (of type Lorenz) known as the Chen system. Vanecek and Celikovsky in [19], characterize a generalized Lorenz system by a condition on its linear part matrix $J = (a_{ij})$: $a_{12}a_{21} > 0$. Lü in [6] observes that while the classical Lorenz system belongs to this class, the Chen system satisfies a *dual* condition $a_{12}a_{21} < 0$, (systems that satisfy such a condition are called dual systems to the Lorenz system) and introduces a new system that bridges the gap between the Chen system and Lorenz system, satisfying $a_{12}a_{21} = 0$. A nonlinear system arising from a nuclear spin generator and compared with the Lorenz system is studied in [12]. Nonlinear dynamics is met in many areas from Ecology [16] to Physics [7].

From the point of view of the potential applications, systems with sensitivity to the initial conditions can be used in secure communications [1], [15]. Of the pioneering papers which proposed to use the chaotic systems in communications are the papers of Pecora and Carrol [10], [11]. Consequently, an appropriate chaotic system can be chosen from a catalogue of chaotic systems to optimize some desirable factors, idea suggested in [15].

These some ideas led us to study a new 3D polynomial differential system given by:

$$\dot{x} = a(y - x), \quad \dot{y} = (c - a)x - axz, \quad \dot{z} = -bz + xy, \quad (2)$$

^{*}Department of Mathematics, "Politehnica" University of Timisoara, P-ta Victoriei, Nr.2, 300006, Timisoara, Timis, Romania, email: gheorghe.tigan@mat.upt.ro

[†]Department of Mathematics, West University of Timisoara, B-dul V. Parvan, Nr. 4, 300223, Timisoara, Timis, Romania, email: opris@math.uvt.ro

with a, b, c real parameters and $a \neq 0$. Call it the T system. Some results regarding the T system are already presented in [17] and [18]. Compared with the Lü system introduced in [6], the system T allows a larger possibility in choosing the parameters of the system and, therefore, it displays a more complex dynamics.

The paper is organized as follows. In the second Section we give some details of the equilibria. In Section 3, we pay attention to the chaos in the system T and, using a method of type Shilnikov, we prove that the system displays "horseshoe" chaos. This implies that the system possesses a strange chaotic attractor. Not any strange dynamics is chaotical as it is pointed out, for example, in [8], where is presented a scenario for a 2-dimensional dynamic system to possess strange nonchaotic behavior.

2 The equilibrium points of the system T

Straightforward computations lead us to the following result:

PROPOSITION 2.1. *If $\frac{b}{a}(c - a) > 0$, the system T possesses three equilibrium isolated points:*

$$O(0, 0, 0), E_1(\sqrt{\frac{b}{a}(c - a)}, \sqrt{\frac{b}{a}(c - a)}, \frac{c-a}{a}), E_2(-\sqrt{\frac{b}{a}(c - a)}, -\sqrt{\frac{b}{a}(c - a)}, \frac{c-a}{a}),$$

and for $b \neq 0, \frac{b}{a}(c - a) \leq 0$ it has only one isolated equilibrium point, $O(0, 0, 0)$.

As reported in [17], we have:

THEOREM 2.1. *For $b \neq 0$ the following statements are true:*

- a) *If $(a > 0, b > 0, c \leq a)$, then $O(0, 0, 0)$ is asymptotically stable ,*
- b) *If $(b < 0)$ or $(a < 0)$ or $(a > 0, c > a)$, then $O(0, 0, 0)$ is unstable.*

THEOREM 2.2. *If $(a + b > 0, ab(c - a) > 0, b(2a^2 + bc - ac) > 0)$, the equilibrium points*

$E_{1,2}(\pm\sqrt{\frac{b}{a}(c - a)}, \pm\sqrt{\frac{b}{a}(c - a)}, \frac{c-a}{a})$ are asymptotically stable.

3 Chaos in the T system via the heteroclinic Shilnikov method

Computing the Lyapunov exponents with the software *Dynamics* [9] for the parameter vector $(a, b, c) = (2.1, 0.6, 30)$ and the initial conditions $(0.1, -0.3, 0.2)$, we get that $\lambda_1 = 0.37 > 0$, $\lambda_2 = 0.00$ and $\lambda_3 = -3.07$. So the system T displays chaotic characteristics.

In the following, using the Shilnikov heteroclinic method, we show that the system T presents chaos of *horseshoe* type. We will employ the following result:

THEOREM 3.1. ([14, 21]) *If a 3D given system $\dot{x} = F(x)$ has two equilibrium points E_1, E_2 , of type saddle-focus, i.e. the eigenvalues of the Jacobi matrix associated to the system in these points are $\gamma_k \in \mathbb{R}$ and $\alpha_k \pm i\beta_k \in \mathbb{C}$, $k = 1, 2$, such that*

$$\alpha_1\alpha_2 > 0 \text{ or } \gamma_1\gamma_2 > 0 \tag{3}$$

and (the Shilnikov inequality)

$$|\gamma_k| > |\alpha_k|, k = 1, 2, \tag{4}$$

and if the system has a heteroclinic orbit connecting the equilibrium points E_1 and E_2 , then the Poincaré map defined on a transversal section of the flow in a neighborhood of the heteroclinic orbit presents chaos of horseshoe type.

We will show that our system T given by

$$\dot{x} = a(y - x), \quad \dot{y} = (c - a)x - axz, \quad \dot{z} = -bz + xy, \quad (5)$$

fulfills the conditions from the above theorem. The equilibrium points of the system T are: $E_{1,2}(\pm\sqrt{\frac{b}{a}(c-a)}, \pm\sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a})$ for $\frac{b}{a}(c-a) > 0$. The characteristic polynomial associated to the Jacobi matrix of the system T in these points is:

$$f(\lambda) = \lambda^3 + \lambda^2(a+b) + bc\lambda + 2ab(c-a) = 0 \quad (6)$$

Denoting $\lambda = \mu - (a+b)/3$, (6) leads to:

$$\mu^3 + p\mu + q = 0, \quad (7)$$

where

$$p = bc - \frac{1}{3}a^2 - \frac{2}{3}ab - \frac{1}{3}b^2 \quad (8)$$

and

$$q = \frac{2}{27}a^3 - \frac{16}{9}a^2b + \frac{2}{9}ab^2 + \frac{2}{27}b^3 + \frac{5}{3}bca - \frac{1}{3}b^2c. \quad (9)$$

Denote $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$. Then, if $\Delta > 0$, Eq. (7) has a negative solution, α_1 , together with a pair of complex solutions, $\alpha_2 \pm i\alpha_3$, where

$$\begin{aligned} \alpha_1 &= \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}, \\ \alpha_2 &= \frac{-1}{2} \left(\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} \right), \\ \alpha_3 &= \frac{\sqrt{3}}{2} \left(\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} \right). \end{aligned}$$

So, if $\Delta > 0$, the three roots of Eq. (6) are:

$$\lambda_1 = -\frac{a+b}{3} + \alpha_1, \quad \lambda_2 = -\frac{a+b}{3} + \alpha_2 + i\alpha_3, \quad \lambda_3 = -\frac{a+b}{3} + \alpha_2 - i\alpha_3, \quad (10)$$

with $\lambda_1 < 0$. From the first two Eqs. of (5) one gets

$$y = x + \frac{\dot{x}}{a},$$

$$z = -\frac{\dot{y} - (c-a)x}{ax} = -\frac{\ddot{x} + a\dot{x}}{a^2x} + \frac{c-a}{a}, \text{ and using the last Eq. of (5) we get}$$

$$\frac{d}{dt} \left(\frac{\ddot{x} + a\dot{x}}{x} \right) + b \frac{\ddot{x} + a\dot{x}}{x} + a^2x^2 + ax\dot{x} - ab(c-a) = 0,$$

or, equivalently

$$x\ddot{x} + (a+b)x\ddot{x} - \dot{x}\ddot{x} - a\dot{x}^2 + abx\dot{x} + a^2x^4 + ax^3\dot{x} - ab(c-a)x^2 = 0. \quad (11)$$

We observe that if $x(t)$ is determined, then we can find $y(t), z(t)$. Therefore, we have to find a function $\phi(t)$ such that $x(t) = \phi(t)$ to satisfy Eq. (11) and $\phi(t) \rightarrow -\sqrt{\frac{b}{a}(c-a)}$ for $t \rightarrow +\infty$,

$\phi(t) \rightarrow \sqrt{\frac{b}{a}(c-a)}$ for $t \rightarrow -\infty$ or, inversely $\phi(t) \rightarrow \sqrt{\frac{b}{a}(c-a)}$ for $t \rightarrow +\infty$, $\phi(t) \rightarrow -\sqrt{\frac{b}{a}(c-a)}$ for $t \rightarrow -\infty$.

We can define, without losing the generality, that the direction from E_1 to E_2 corresponds to $t \rightarrow \infty$, and from E_2 to E_1 corresponds to $t \rightarrow -\infty$.

Consider $\phi(t)$ of the following form $\phi(t) = -x_0 + \sum_{n=1}^{\infty} a_n e^{n\alpha t}$, with α real number and $x_0 = \sqrt{\frac{b}{a}(c-a)}$.

Identifying the coefficients of $e^{n\alpha t}$ in (11) we get:

$$x_0 [\alpha^3 + \alpha^2(a+b) + bc\alpha + 2ab(c-a)] a_1 = 0, \quad (12)$$

$$a_2 = \frac{1}{f(2\alpha)x_0} (\alpha^2 b + ab\alpha - ab(c-a) + 6a^2 x_0^2 + 3\alpha a x_0^2) a_1^2, \quad (13)$$

$$a_3 = -\frac{1}{f(3\alpha)x_0} (4a^2 x_0 a_1^3 + 3\alpha a x_0 a_1^3) + \frac{1}{f(3\alpha)x_0} [3\alpha^3 + \alpha^2 a + 5\alpha^2 b + 3ab\alpha - 2abc + 2a^2 b + 12a^2 x_0^2 + 9\alpha a x_0^2] a_1 a_2, \quad (14)$$

and for $n \geq 4$

$$a_n = \frac{1}{f(n\alpha)x_0} (a^2 a_{ijpq} + ab_{ijpq}) + \frac{1}{f(n\alpha)x_0} \sum_{i+j=n} \alpha^3 (j^3 - ij^2) a_i a_j + \frac{1}{f(n\alpha)x_0} \sum_{i+j=n} [\alpha^2 ((a+b)j^2 - ija) + ab\alpha j - ab(c-a) + 6a^2 x_0^2 + 3\alpha a j x_0^2] a_i a_j, \quad (15)$$

where

$$f(n\alpha) = (n\alpha)^3 + (n\alpha)^2(a+b) + bc n\alpha + 2ab(c-a), \quad (16)$$

$$a_{ijpq} = -4x_0 \sum_{i+j+p=n} a_i a_j a_p + \sum_{i+j+p+q=n} a_i a_j a_p a_q, \quad (17)$$

$$b_{ijpq} = -3\alpha x_0 \sum_{i+j+p=n} p a_i a_j a_p + \alpha \sum_{i+j+p+q=n} p a_i a_j a_p a_q, \quad (18)$$

with $i, j, p, q \geq 1$.

In order to can be implemented on the computer, we put the sums (17) and (18) in the following forms:

$$a_{ijpq} = -4x_0 \sum_{i=1}^{n-2} a_i \left(\sum_{j=1}^{n-i-1} a_j a_{n-i-j} \right) + \sum_{i=1}^{n-3} a_i \left(\sum_{j=1}^{n-i-1} a_j \left(\sum_{p=1}^{n-i-j-1} a_p a_{n-i-j-p} \right) \right), \quad (19)$$

$$b_{ijpq} = -3\alpha x_0 \sum_{i=1}^{n-2} a_i \left(\sum_{j=1}^{n-i-1} (n-i-j) a_j a_{n-i-j} \right) + \sum_{i=1}^{n-3} a_i \left(\sum_{j=1}^{n-i-1} a_j \left(\sum_{p=1}^{n-i-j-1} p a_p a_{n-i-j-p} \right) \right). \quad (20)$$

Assume $a_1 \neq 0$. If not, we observe inductively that all coefficients $a_n, n \geq 2$, are zero. Hence, from (12), we get

$$\alpha^3 + \alpha^2(a+b) + bc\alpha + 2ab(c-a) = 0 \quad (21)$$

that is α is the negative root of characteristic polynomial (6). Because Eq. (6) has a single negative solution for $\Delta > 0$, we get that

$$f(n\alpha) = (n\alpha)^3 + (n\alpha)^2(a+b) + bc n\alpha + 2ab(c-a) \neq 0, n > 1. \quad (22)$$

Consequently, the coefficients a_n are completely determined by a, b, c, α and a_1 , and they are of the following form:

$$a_n = g(n)a_1^n, n > 1 \quad (23)$$

where the terms $g(n)$ are known functions. So the corresponding branch of the heteroclinic orbit for $t > 0$ is determined. In a similar manner, consider the case $t < 0$ and $\phi(t) = x_0 + \sum_{n=1}^{\infty} a_n e^{-n\beta t}$, with β real, and find $\beta = \alpha$ and $b_n = -a_n, n > 0$. Consequently, the heteroclinic orbit $\phi(t)$ is given by:

$$\phi(t) = \begin{cases} -x_0 + \sum_{n=1}^{\infty} a_n e^{n\alpha t} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ x_0 - \sum_{n=1}^{\infty} a_n e^{-n\alpha t} & \text{for } t < 0 \end{cases} \quad (24)$$

Following a method presented in [21] one can show that $\phi(t)$ is uniformly convergent. So we have the following result that characterizes the chaos in the system T :

THEOREM 3.2. *If $\Delta > 0$ and $\alpha_1 + \alpha_2 < -\frac{2(a+b)}{3}$, then the system T has a heteroclinic orbit given by (24), which connects the equilibrium points E_1, E_2 , so the chaos is of horseshoe type.*

Let us partially illustrate the particular case $a = 2.1, b = 0.6, c = 30$. Then $\lambda_1 = \alpha = -3.429$, $\lambda_2 = 0.364 - 4.513i, \lambda_3 = 0.364 + 4.513i, \Delta = 911.69 > 0, x_0 = 2.8234$. Observe that the equilibrium points E_1, E_2 are saddle-focus. Imposing that $\phi(t)$ to be at least continue, from $\phi(0_-) = \phi(0_+) = 0$, we find the equation

$$-x_0 + a_1 + a_2 + \dots + a_n = 0, n > 1. \quad (25)$$

Solving this equation, we observe that the first coefficient is $a_1 = 3.051$ for any $n > 10$.

REMARK 3.1. *Numerical series (24) which describes the heteroclinic orbit between E_1 and E_2 , is rapidly convergent. For example, considering the first ten terms of the series, for $t = 10$ we get*

$$\phi(t) = -x_0 + \sum_{n=1}^{10} a_n e^{n\alpha t} = -2.8234, \text{ for } t = -10, \phi(t) = x_0 - \sum_{n=1}^{10} a_n e^{-n\alpha t} = 2.8234 \text{ and for } t = 0,$$

$$\phi(t) = -x_0 + \sum_{n=1}^{10} a_n e^{n\alpha t} = 0.0000. \text{ So the equilibrium points } E_1, E_2 \text{ and } O \text{ are found with an}$$

approximation of four exact decimals because $x_0 = \sqrt{\frac{b}{a}(c-a)} = 2.8234$.

4 Conclusions

In this paper we investigated a new chaotic system. For some parameter vectors, the system presents the property of dependence to initial conditions which is a necessary condition for a system to be chaotic. The system possesses three equilibrium points, the origin $O(0, 0, 0)$ and another two points $E_{1,2}$. The chaos of horseshoe type was analyzed using the heteroclinic Shilnikov method, constructing a heteroclinic connection between the saddle-focus equilibrium points E_1 and E_2 .

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